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On the Closability of Some Positive Definite Symmetric Differential Forms on $C_0^\infty(\Omega)$

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Let $\mathscr{E}$ be a Dirichlet form in $L^2(\Omega; m)$, where Ω is an open subset of $\mathbb{R}^n$, $n \geqslant 2$, and m a Radon measure on Ω; for each integer k with $1 \leqslant k < n$, let $\mathscr{E}_k$ be a Dirichlet form on some k-dimensional submanifold Ω_k of Ω. The paper is devoted to the study of the closability of the forms E with domain $C_0^\infty(\Omega)$ and defined by: $E(f, g) = \mathscr{E}(f, g) + \sum_{i=1}^{p} \mathscr{E}_{k_i}(f^{k_i}, g^{k_i})$, where $1 \leqslant k_p < \cdots < n$, and where f^{k_i}, g^{k_i} denote restrictions of f, g in $C_0^\infty(\Omega)$ to Ω_{k_i}. Conditions are given for E to be closable if, for each $i = 1,..., p$, one has $k_i = n - i$. Other conditions are given for E to be non-closable if, for some i, $k_i < n - i$. © 1985 Academic Press, Inc.

1. INTRODUCTION

(a) Let Ω be a non-empty open subset of $\mathbb{R}^n$, $n \geqslant 2$, and let m be a positive Radon measure supported by Ω. A family $(a_{i,j})_{1 \leqslant i,j \leqslant n}$ of real locally integrable functions on Ω such that, for almost all x in Ω, $(a_{i,j}(x))_{i,j}$ is positive definite symmetric $n \times n$ matrix gives rise to a positive symmetric differential form $\mathscr{E}$ on $C_0^\infty(\Omega)$ such that:

$$\mathscr{E}(f, g) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial f}{\partial x_i}(x) \cdot \frac{\partial g}{\partial x_j}(x) \cdot a_{i,j}(x)\, dx. \tag{1.1}$$

We keep the same notation for the closure of this form in the real $L^2(\Omega; m)$

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space whenever this closure exists. The closability problem for the forms (1.1) has been extensively discussed in [1, 3, 6, 7].

Now suppose there are other forms $\mathscr{E}_\alpha$ of this kind living in smooth submanifolds Ω_α of Ω, α running on a finite set, and let f^α be the restriction to Ω_α of the element f in $C_0^\infty(\Omega)$. In this note we are concerned with the closability problem for the forms $\mathscr{E}_\alpha$ and their sums in the real Hilbert space $L^2(\Omega; dx)$, dx being the Lebesgue measure in Ω.

(**b**) Although this problem may be of independent interest, we are discussing it with particular application in mind, namely, in the study of the irreducibility of the energy representations of a gauge group (see, e.g., [2, 4]). The study of the irreducibility of such a representation π is closely connected with deep properties of the standard Gaussian measure μ_π on the distribution space $\mathscr{D}'(\Omega)$ with Fourier transform $\hat{\mu}_\pi$ given by

$$\hat{\mu}_\pi(f) = \exp\{-\tfrac{1}{2} E_\pi(f, f)\}, \qquad f \in C_0^\infty(\Omega),$$

where E_π is a particular Dirichlet form associated to π. In the case $E_\pi = \mathscr{E}$, the problem was solved in [2]. Recently, using ideas developed in [4], one of us (J. Marion) and D. Testard studied energy representations π for which

$$\hat{\mu}_\pi(f) = \exp\left\{-\frac{1}{2}\mathscr{E}(f, f) - \frac{1}{2}\sum_\alpha \mathscr{E}_\alpha(f^\alpha, f^\alpha)\right\} \qquad [5].$$

The only method known up to now for studying this problem of irreducibility is that developed in Ref. [2]; however, the use of this method is subordinate to the closability of the differential form given by

$$E(f, f) = \mathscr{E}(f, f) + \sum_\alpha \mathscr{E}_\alpha(f^\alpha, f^\alpha).$$

Since out motivation comes from the above problem, we shall not make much effort at being very general but only to the extent necessary for the application.

2. Notations and Results

(**a**) Let $n \geqslant 2$, let $\Omega = \Omega_n$ be a non-empty open subset of $\mathbb{R}^n$, and let us select $n-1$ constants $c_2, ..., c_n$ such that, for all integers s with $1 \leqslant s \leqslant n-1$,

$$\Omega_s = \{(X_1, ..., X_s, ..., X_n) \in \Omega / X_{s+1} = c_{s+1}, ..., X_n = c_n\} \neq \varnothing;$$

an element $(X_1, \ldots, X_s, c_{s+1}, \ldots, c_n)$ of Ω_s will be denoted X^s, and the Lebesgue measure in Ω_s will be denoted dX^s. We shall consider also the function $\eta_s : \Omega_s \to \mathbb{R}$ such that $\eta_s(Y^s)$ is the distance from the point Y^s to the set Ω_{s-1}. We shall denote by $\mathcal{M}^s$ the family of $s \times s$ symmetric matrices $A_s = (a^s_{i,j})$ of $L^1_{\text{loc}}(\Omega_s; dX^s)$ functions $a^s_{i,j}$ such that, for almost all X^s in Ω_s, $A^s(X^s)$ is positive definite. Each element $A^s = (a^s_{i,j})$ in $\mathcal{M}^s$ gives rise to a positive definite symmetric differential form $\mathscr{E}_s$ defined, for f, g in $C_0^\infty(\Omega_s)$, by

$$\mathscr{E}_s(f, g) = \sum_{i,j=1}^{s} \frac{\partial f(X^s)}{\partial X_i} \cdot \frac{\partial g(X^s)}{\partial X_j} a^s_{i,j}(X^s)\, dX^s. \tag{2.1}$$

DEFINITION 1. Let s be such that $1 \leqslant s \leqslant n$. We shall say that the element A^s in $\mathcal{M}^s$ has the property (π_1) if $\mathscr{E}_s$ is closable in $L^2(\Omega_s; dX^s)$.

DEFINITION 2. Let s be such that $2 \leqslant s \leqslant n$, and let A^s be in $\mathcal{M}^s$. We shall say that A^s has the property (π_2) if there are an open subset U_s in $\mathbb{R}^n$ with $\Omega_{s-1} \subset U_s \cap \Omega_s$, a real number α satisfying $-1 < \alpha < 1$, and a non-negative $L^1_{\text{loc}}(\Omega_s; dX^s)$ function λ_s such that:

(i) either for all X^s in $U_s \cap \Omega_s$ with $X_s > c_s$, or for all X^s in $U_s \cap \Omega_s$ with $X_s < c_s$, $\lambda_s(X^s) = (\eta_s(X^s))^\alpha$;

(ii) let Λ_s be the symmetric form defined on $C_0^\infty(\Omega_s)$ by

$$\Lambda_s(f, g) = \sum_{i,j=1}^{s} \int_{\Omega_s} \frac{\partial f(X^s)}{\partial X_i} \cdot \frac{\partial g(X^s)}{\partial X_j} \lambda_s(X^s)\, dX^s; \tag{2.2}$$

then, for all f in $C_0^\infty(\Omega_s)$,

$$\mathscr{E}_s(f, f) \geqslant \Lambda_s(f, f). \tag{2.3}$$

Now, after recalling that for f in $C_0^\infty(\Omega)$, f^s denotes the restriction of f to Ω_s, we are ready to formulate the first result.

THEOREM 1. *Let k be an integer such that $1 \leqslant k \leqslant n-1$; if for all s with $k \leqslant s \leqslant n$, the forms $\mathscr{E}_s$ defined by* (2.1) *satisfy properties* (π_1) *and* (π_2), *the symmetric form $\mathscr{E}_{n,k}$ defined in $C_0^\infty(\Omega)$ by*

$$\mathscr{E}_{n,k}(f, g) = \mathscr{E}_n(f, g) + \sum_{j=1}^{n-k} \mathscr{E}_{n-j}(f^{n-j}, g^{n-j}) \tag{2.4}$$

is closable in $L^2(\Omega; dX)$.

Remark. Note that Theorem 1 is not applicable if in the sum (2.4) a form $\mathscr{E}_s$ with $k \leqslant s \leqslant n$ vanishes identically.

(**b**) This remark suggests that it may be useful to know the conditions preventing a sum of forms from being closable. For this we shall need the concept of capacity (see Ref. [3]). Here we have a less general definition and some simple facts about capacities.

Let $\mathcal{O}$ be the family of open subsets of Ω, and for A in $\mathcal{O}$ we put

$$\mathcal{L}_A = \{f \in D(\mathcal{E}_n)/f(X) \geqslant 1 \text{ for almost all } X \text{ in } A\},$$

where $D(\mathcal{E}_n)$ denotes the domain of $\mathcal{E}_n$.

DEFINITION 3. The capacity of an element A of $\mathcal{O}$ is defined by

$$\begin{aligned} \operatorname{cap}(A) &= \infty && \text{if} \quad \mathcal{L}_A = \varnothing \\ &= \inf_{f \in \mathcal{L}_A} \left(\mathcal{E}_n(f, f) + \int_\Omega |f|^2 \, dx\right) && \text{otherwise,} \end{aligned}$$

and for any subset B of Ω we put

$$\operatorname{cap}(B) = \inf_{\substack{A \in \mathcal{O} \\ B \subset A}} \operatorname{cap}(A).$$

Clearly $\operatorname{cap}(A) \geqslant l(A)$, where $l(A)$ stands for the Lebesgue measure of A, and in fact a set of measure zero may have strictly positive capacity.

If for instance $n = 1$, then a one-point set $\{X_0\}$ has a non-zero capacity unless $a_{11}(X)$ converges sufficiently fast to zero as X tends to X_0, where sufficiently fast means a behavior like $|X - X_0|^\alpha$ with $\alpha \geqslant 1$. If $a_{11}(X)$ behaves like $|X - X_0|^\alpha$ with $-1 < \alpha < 1$, then $\operatorname{cap}\{X_0\}$ is finite and strictly positive; if $\alpha \leqslant -1$, then $\operatorname{cap}\{X_0\}$ is finite and strictly positive; if $\alpha \leqslant -1$, $\operatorname{cap}\{X_0\} = \infty$. When $n > 1$, and the symmetric form is given by (2.2), the condition $\lambda_n(X) = (\eta_n(X))^\alpha$ with $-1 < \alpha < 1$ guarantees a strictly positive capacity for Ω_{n-1}; if moreover Ω_{n-1} is bounded then $0 < \operatorname{cap}(\Omega_{n-1}) < +\infty$.

However, with the same condition the sets Ω_s with $1 \leqslant s \leqslant n-2$ have vanishing capacities. Explanation of the above facts can be found in Ref. [3] (which gives other numerous references).

THEOREM 2. *Let $\mathcal{E}_s$ be the forms defined by* (2.1), $1 \leqslant s \leqslant n$, *and satisfying property* (π_1). *Let us assume moreover that for some integer l with $n \geqslant l > k$ and all integer m such that $n \geqslant m \geqslant l$ the following conditions are fulfilled*:

(i) *The diagonal forms $\mathcal{E}_d^m$ given for all f, g in $C_0^\infty(\Omega_m)$ by*

$$\mathcal{E}_d^m(f, g) = \sum_{i=1}^m \int_{\Omega_m} \frac{\partial f}{\partial X_i} \cdot \frac{\partial g}{\partial X_i} \, p_i^m(X^m) \, dX^m$$

are closable in $L^2(\Omega_m : dX^m)$ *and for all* f *in* $C_0^\infty(\Omega)$

$$\mathscr{E}_d^m(f^m, f^m) \geqslant \mathscr{E}_m(f^m, f^m); \tag{2.5}$$

(ii) *there exists an open subset* U_l *in* Ω *with* $\Omega_{l-1} \cap U_l \neq \varnothing$ *and with* $\mathrm{cap}_{d,m}(\Omega_{l-1} \cap U_l) = 0$, *where* $\mathrm{cap}_{d,m}$ *denotes the capacity with respect to the form* $\mathscr{E}_d^m$ *in* $L^2(\Omega_m; dX^m)$;

(iii) *there is* m' *with* $k \leqslant m' < l$ *such that* $\mathscr{E}_{m'}$ *is not identically zero.*

Then, the form $\mathscr{E}_{n,k}$ *defined by* (2.4) *is not closable in* $L^2(\Omega; dX)$.

3. Proofs of Theorems 1 and 2

(**a**) We begin with the proof of Theorem 1. The main tool is a simplified version of the imbedding theorem for the weighted Sobolev spaces $W_p^m(\Omega; (\eta(X))^\alpha)$ of Ref. [8, Sect. 3.6.1].

Within the assumptions of Theorem 1, that is to say properties (π_1) and (π_2), it implies the existence of a positive constant c so that for any integer l fulfilling $k < l \leqslant n$, and all f in $C_0^\infty(\Omega)$, $(A_l(f^l, f^l))^{1/2} \geqslant c \|\mathscr{F}^{-1}(1 + \|X^{l-1}\|^2)^{1/2} \cdot \mathscr{F} f^{l-1}\|_{L^2(\Omega_{l-1})}$, where $\mathscr{F}$ denotes the Fourier transform, and $s = (1-\alpha)/2$. As the right-hand side is greater than $C\|f^{l-1}\|_{L^2(\Omega_{l-1})}$, we have

$$(A_l(f^l, f^l))^{1/2} \geqslant c \|f^{l-1}\|_{L^2(\Omega_{l-1})}. \tag{3.1}$$

Suppose that a sequence $(f_\nu)_\nu$ in $C_0^\infty(\Omega)$ converges to zero in $L^2(\Omega; dX)$ and that

$$\lim_{\substack{\nu \to +\infty \\ \mu \to \infty}} \mathscr{E}_{n,k}(f_\nu - f_\mu, f_\nu - f_\mu) = 0.$$

From the closability of $\mathscr{E}_n$ it follows that $\lim_{\nu \to \infty} \mathscr{E}_n(f_\nu, f_\nu) = 0$ and hence $A_n(f_\nu, f_\nu)$ converges to zero. This and (3.1) imply that (f_ν^{n-1}) converges to zero in $L^2(\Omega_{n-1})$, so that $\lim_{\nu \to \infty} \mathscr{E}_{n-1}(f_\nu^{n-1}, f_\nu^{n-1}) = 0$. The same argument applied to $\mathscr{E}_{n-2}, \ldots, \mathscr{E}_k$ completes the proof of Theorem 1.

(**b**) In order to prove Theorem 2, we need two lemmas.

Lemma 3.1. *Let* Ω *be a bounded open subset in* $\mathbb{R}^n$, *let* s *be an integer such that* $1 \leqslant s \leqslant n$, *let* φ_i *and* ψ_i, $1 \leqslant i \leqslant s$, *be* $2s$ *piecewise continuous functions in* $L^2(\Omega; dX)$ *and let us assume that for any open subset* U *in* Ω *the integral* $\int_U \psi_i^2(X)\, dX > 0$ *for* $i = 1, 2, \ldots, s$.

Let $(f_\nu)_\nu = (f_\nu^1, \ldots, f_\nu^i, \ldots, f_\nu^s)_\nu$ *be a sequence in* $C_0^\infty(\Omega)^s$ *such that:*

(a) $\sum_{i=1}^{s} \int_\Omega |f_v^i(X)|^2 \cdot \psi_i^2(X)\, dX \to 0$ *as* $v \to \infty$;

(b) *there is a constant* $C > 0$ *such that* $|f_v^i(X)| < C$ *for all* $i = 1,..., s$, *all* v, *all* X *in* Ω;

then $\sum_{i=1}^{s} \int_\Omega |f_v^i(X)|^2 \cdot \varphi_i^2(X)\, dX \to 0$ as $v \to \infty$.

Proof. Let $V_i = \{X \in \Omega / \psi_i(X) = 0\}$, $i = 1,..., s$. Given $\varepsilon > 0$, we can find an open neighbourhood U_i of V_i such that

$$\int_{U_i} |f_v^i(X)|^2\, \varphi_i^2(X)\, dX \leqslant C \int_{U_i} \varphi_i^2(X)\, dX < \frac{1}{2}\varepsilon.$$

Thus

$$\int_\Omega |f_v^i(X)|^2\, \varphi_i^2(X)\, dX$$

$$< \frac{1}{2}\varepsilon + \int_{\Omega - U_i} |f_v^i(X)|^2 \cdot \frac{\varphi_i^2(X)}{\psi_i^2(X)}\, \psi_i^2(X)\, dX$$

$$< \frac{1}{2}\mathscr{E} + \left(\int_{\Omega - U_i} |f_v^i(X)|^2\, \psi_i^2(X)\, dX \cdot \int_{\Omega - U_i} \frac{\varphi_i^2(x)}{\psi_i^2(X)}\, dX\right)^{1/2} < \varepsilon$$

for v sufficiently large.

(c) Let us suppose here that $n \geqslant 3$, and let P, P' be two open cubes in $\mathbb{R}^n$ such that $\bar{P} \subset P'$, $\overline{P'} \subset \Omega$. We can write

$$P = \prod_{i=1}^{n} \,]a_i, b_i[, \qquad P' = \prod_{i=1}^{n} \,]a_i', b_i'[$$

with $a_i' < a_i < b_i < b_i'$, $i = 1, 2,..., n$, and where the constant c_i that appears in definition of Ω_{i-1} is such that $a_i < c_i < b_i$.

LEMMA 3.2. *Let* $\mathscr{E}_d^n$ *be a diagonal form defined on* $C_0^\infty(\Omega)$ *which is closable in* $L^2(\Omega, dX)$, *let* s *be such that* $1 \leqslant s \leqslant n-1$, *and* $\operatorname{cap}_{d,n}(\Omega_s) = 0$. *For any pair of cubes* P, P' *as described above there exist*:

(i) *a function* φ *in* $C_0^\infty(\mathbb{R}^s)$ *with* $\operatorname{supp}(\varphi) \subset \prod_{i=1}^{s} \,]a_i', b_i'[$, $\varphi(X^s) = 1$ *for all* X^s *in* $\prod_{i=1}^{s} \,]a_i, b_i[$ *and such that* $0 \leqslant \varphi \leqslant 1$;

(ii) *a sequence* $(\psi_v)_v$ *of functions* ψ_v *in* $C_0^\infty(\mathbb{R}^{n-s})$ *with* $\operatorname{supp}(\psi_v) \subset \prod_{i=s+1}^{s} \,]a_i', b_i'[$, $0 \leqslant \psi_v \leqslant 1$, $\psi_v(c_{s+1},..., c_n) = 1$.

Moreover, if we put $(X_{s+1},..., X_n) = Y^{n-s}$ *and* f_v: $X = (X_1,..., X_s,..., X_n) \to \varphi(X_1,..., X_s)\, \psi_v(Y^{n-s})$, *then*

$$\lim_{v \to \infty} \left\{\mathscr{E}_d^n(f_v, f_v) + \int_\Omega |f_v|^2\, dX\right\} = 0.$$

Proof. Let P, P' be the given cubes. Since $\mathrm{cap}_{d,n}(\Omega_s)=0$, the monotonicity of capacities implies $\mathrm{cap}_{d,n}(\Omega_s \cap P)=0$. By Ref. [1] the capacity of $\Omega_s \cap P$ is also zero if $\mathscr{E}_d^n$ is restricted to $L^2(P', dX)$. Thus we can find a sequence $(g_\nu)_\nu$ in $C_0^\infty(P')$ which converges to zero in the $\mathscr{E}_d^n$-norm, and such that $0 \leqslant g_\nu \leqslant 1$ and $g_\nu(X)=1$ for all X in $P \cap \Omega_s$. Our aim is to prove that such functions can be given by the products $\varphi\psi_\nu$ which appear in part (ii) of the lemma. Let $(g_\nu)_\nu$ be a sequence converging to zero in the $\mathscr{E}_d^n$-norm; then, even if we replace the sequence $(g_\nu)_\nu$ by a subsequence, we can assume that

$$\int_{a'_{s+1}}^{b'_{s+1}} \cdots \int_{a'_n}^{b'_n} g_\nu^2(X_1,..., X_s, Y^{n-s})\, dY^{n-s} \to 0$$

and

$$\sum_{i=s+1}^{n} \int_{a'_{s+1}}^{b'_{s+1}} \cdots \int_{a'_n}^{b'_n} \left(\frac{\partial g_\nu}{\partial X_i}\right)^2 (X_1,..., X_s, Y^{n-s})\, dY^{n-s} \to 0$$

as $\nu \to \infty$, where $(X_1,..., X_s) \in \prod_{i=1}^{s}\,]a_i, b_i[$. Since there is no loss of generality to assume that these functions are piecewise continuous and different from zero off Ω_s the pairs of functions

$$(X_1,..., X_s, Y^{n-s}) \to p_i^2(X_1,..., X_s, Y^{n-s})$$

and

$$(X_1,..., X_s, Y^{n-s}) \to \int_{a'_1}^{b'_1} \cdots \int_{a'_s}^{b'_s} \varphi^2(X_1,..., X_s)\, p_i^2(X_1,..., X_s, Y^{n-s})\, dX^s$$

satisfy condition of Lemma 3.1. It follows that

$$\lim_{\nu\to\infty} \left\{ \sum_{i=s+1}^{n} \int_{a'_1}^{b'_1} \cdots \int_{a'_n}^{b'_n} \left[\left(\frac{\partial f_\nu}{\partial X_i}\right)^2 p_i^2(X) + f_\nu^2 \right] dX \right\} = 0,$$

where $f_\nu\colon (X_1,..., X_s,..., X_n) \to \varphi(X^s)\, g_\nu(X_1,..., X_s, Y^{n-s})$ with, here, $X^s = (X_1,..., X_s)$.

Observe now that

$$\sum_{i=1}^{s} \int_{a'_1}^{b'_1} \cdots \int_{a'_s}^{b'_s} \left(\frac{\partial \varphi}{\partial X_i}\right)^2 (X^s)\, p_i^2(X)\, dX^s$$

defines still another measure on $\prod_{i=s+1}^{n}\,]a'_i, b'_i[$, and then, again by Lemma 3.1,

$$\lim_{\nu\to+\infty} \left\{ \sum_{i=1}^{s} \int_{a'_1}^{b'_1} \cdots \int_{a'_n}^{b'_n} \left(\frac{\partial f_\nu}{\partial X_i}\right)^2 (X)\, p_i^2(X)\, dX \right\} = 0.$$

This completes the proof of the fact that the sequence $(f_\nu)_\nu$ with $f_\nu\colon X \to \varphi(X^s)\, g_\nu(X^s, Y^{n-s})$ converges to zero in the $\mathscr{E}^n_d$-norm.

(**d**) We are able now to prove Theorem 2. Without loss of generality we can take

$$U_l = \prod_{i=1}^{l} \,]a'_i, b'_i[,$$

with a'_i, b'_i as in Lemma 3.2; let $(f_\nu)_\nu$ be the sequence of this lemma with $s = l-1$.

From the above results if follows that

$$\lim_{\nu\to\infty} \{\mathscr{E}_m(f^m_\nu, f^m_\nu)\} = 0 \qquad \text{for} \quad n \geqslant m \geqslant l.$$

Moreover $\mathscr{E}_{m'}(f^{m'}_\nu - f^{m'}_\mu, f^{m'}_\nu - f^{m'}_\mu) = 0$, so that $(f^m_\nu)_\nu$ is a Cauchy sequence for the norm $f \to |\,\|f\|\,| = \{\mathscr{E}_{n,k}(f, f) + \|f\|^2_{L^2(\Omega, dX)}\}^{1/2}$. However $\mathscr{E}_{m'}(f^{m'}_\nu, f^{m'}_\nu)$ $=$ constant > 0, and hence $\mathscr{E}_{n,k}$ is not closable.

4. Applications

(**a**) Let us consider the symmetric form $E_{l,k}$ defined on $C_0^\infty(\Omega)$ by

$$E_{l,k}(f, g) = \mathscr{E}_{l-1}(f^{l-1}, g^{l-1}) + \mathscr{E}_{l-2}(f^{l-2}, g^{l-2}) + \cdots + \mathscr{E}_k(f^k, g^k) \tag{4.1}$$

with $n \geqslant l > k$, and $\mathscr{E}_{l-1}$ not vanishing identically. This sum can be completed to (2.4) by putting $\mathscr{E}_n = \cdots \mathscr{E}_l = 0$, and then (2.5) holds for any $\mathscr{E}^m_d$. In particular we can choose p^m convergent to zero as X^m goes to Ω_{l-1} sufficiently fast for $\operatorname{cap}_{d,m}(\Omega_{l-1}) = 0$. Thus Theorem 2 is applicable, and one gets:

Corollary 1. *The form $E_{l,k}$ defined on $C_0^\infty(\Omega)$ by (4.1) with $n \geqslant l > k \geqslant 1$ with $\mathscr{E}_{l-1}$ not vanishing identically is not closable in $L^2(\Omega; dX)$.*

(**b**) Corollary 2. *Let $n \geqslant 2$, and let $\mathscr{E}_n$, $\mathscr{E}_{n-1}$ satisfying property (Π_1). Moreover assume that there exists a positive constant d such that $d\mathbb{1} \leqslant A^n(X)$ for an $a \cdot X$ in Ω, where $\mathbb{1}$ is the unit $n \times n$ matrix, and A^n is the element of $\mathscr{M}^n$ which defines $\mathscr{E}_n$. Then, the form*

$$E_{n,n-1}\colon (f, g) \in C_0^\infty(\Omega)^2 \to \mathscr{E}_{n-1}(f^{n-1}, g^{n-1})$$

is closable in $L^2(\Omega; dX)$.

Proof. If we put

$$\Lambda_n(f, g) = d \cdot \sum_{i,j=1}^{n} \int_\Omega \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} dX,$$

then property (Π_2) is fulfilled for $s = n$, $\alpha = 0$; hence the result follows from Theorem 1.

(**c**) COROLLARY 3. *Let $n \geqslant 3$, let $\mathscr{E}_n$, $\mathscr{E}_{n-2}$ satisfying property (Π_1), and assume that there is a positive constant D such that $A^n(X) \leqslant D\mathbb{1}$ for an $a \cdot X$ in Ω. The form*

$$\mathscr{E}_{n,n-2}: (f, g) \to \mathscr{E}_n(f, g) + \mathscr{E}_{n-2}(f^{n-2}, g^{n-2})$$

defined on $C_0^\infty(\Omega)$ is not closable in $L^2(\Omega; dX)$.

Proof. Let us define

$$\mathscr{E}_d^n: (f, g) \to D \sum_{i=1}^{n} \int_\Omega \frac{\partial f}{\partial X_i} \cdot \frac{\partial g}{\partial X_i} dX,$$

$\mathscr{E}_{n-1} = 0$, and $\mathscr{E}_d^{n-1}$ with $p^{n-1}(X)$ quickly decreasing to zero as X tends to Ω_{n-2}; one gets $\mathrm{cap}_{d,n}(\Omega_{n-2}) = \mathrm{cap}_{d,n-1}(\Omega_{n-2}) = 0$, and then the result follows from Theorem 2.

(**d**) Let again $n \geqslant 3$, and $\mathscr{E}_n$, $\mathscr{E}_{n-2}$ fulfill property (Π_1). Suppose further that there is a diagonal form $\mathscr{E}_{d,n}$ such that, for all f in $C_0^\infty(\Omega)$, $\mathscr{E}_n(f, f) \geqslant \mathscr{E}_{d,n}(f, f)$ and such that for any open subset $U \subset \Omega$ with $U \cap \Omega_{n-2} \neq \varnothing$ we have $\mathrm{cap}_{d,n}(U \cap \Omega_{n-2}) \neq 0$. In this case neither Theorem 1 nor Theorem 2 is applicable to the form $\mathscr{E}_n + \mathscr{E}_{n-2}$. We conjecture that it is closable but we have no proof.

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